

Grand Symmetry, Spectral Action and the Higgs Mass

Agostino Devastato^{1,2}, Fedele Lizzi^{1,2,3} and Pierre Martinetti^{1,2}

¹*Dipartimento di Fisica, Università di Napoli Federico II*

²*INFN, Sezione di Napoli
Monte S. Angelo, Via Cintia, 80126 Napoli, Italy*

³*Departament de Estructura i Constituents de la Matèria,
Institut de Ciències del Cosmos, Universitat de Barcelona,
Barcelona, Catalonia, Spain*

agostino.devastato@na.infn.it, fedele.lizzi@na.infn.it,
martinetti.pierre@gmail.com

Abstract

In the context of the spectral action and noncommutative geometry approach to the standard model, we build a model based on a larger symmetry. This symmetry satisfies all the conditions to have a noncommutative manifold, and mixes gauge and spin degrees of freedom and does not introduce extra fermions. With this *grand* symmetry it is natural to have the scalar field necessary to obtain the Higgs mass in the vicinity of 126 GeV. The spectral action breaks the grand symmetry to the standard model algebra. This breaking also gives the spin structure of spacetime as broken symmetry.

1 Introduction

Noncommutative geometry [1–4] generalizes the concepts of ordinary geometry in an algebraic setting and enables powerful generalizations beyond the Riemannian paradigm. The application of this framework to the standard model of fundamental interactions is a fascinating one [5–8]. In this case, the geometrical setting is that of an usual manifold (spacetime) described by the algebra of complex valued functions defined on it, tensor multiplied by a finite dimensional matrix algebra. This is usually called an “almost commutative geometry”. The standard model is described as a particular almost commutative geometry, and the corresponding Lagrangian is built from the spectrum of a generalized Dirac operator. This noncommutative geometry description of the standard model has a phenomenological predictive power and is approaching the level of maturity which enables it to confront with experiments.

Schematically there are two sides of the application of noncommutative geometry to the standard model. On one side there is the mathematical request that a topological space is a manifold. This yields a set of algebraic requirements [9] involving the algebra of functions defined on the space, represented as bounded operators on a separable Hilbert space, and a (generalized) Dirac operator, plus two more operators representing charge conjugation and chirality. These requirements, being algebraic, can easily be applied to noncommutative algebras. In particular, their application to the almost commutative case singles out the algebra corresponding to the standard model among a restricted number of cases [7, 10] as the smallest algebra which satisfies the requirements.

The other side has to do with the spectral nature of the action. The spectral action principle [11] puts gauge theories, such as the standard model, on the same geometrical footing as general relativity deriving a Lagrangian from a noncommutative spacetime, making it possible unification with gravity. The principle is purely spectral, based on the regularization of the eigenvalues of the Dirac operator* and of its fluctuations, and the action could be derived from its fermionic counterpart via the renormalization flow in the presence of anomalies [17–19].

In [7] (see also [10, 20]) this noncommutative model was enhanced to include massive neutrinos and the seesaw mechanism. The most remarkable result is the possibility to predict the mass of the Higgs particle from the mass of the other fermions and the value of the unification scale, but there are also cosmological predictions based on the spectral action, for example in [21–23].

The prediction of the mass of the Higgs in the vicinity of 126 GeV however depends on the presence of a scalar field, called σ , which was introduced in [24]. In the original paper however this field is qualitatively different from the other bosons. The latter appear as

*The spectral action principle, as well as any finite mode regularization [12–14], requires a Euclidean compact spacetime, but the cutoff on the momentum eigenvalues is even more general and can be used also for continuum spectrum, see for example [15, 16].

elements of the connection one form, i.e. as potentials. This unifies the Higgs with the other vector bosons. The new field introduced in [24] instead does not come from a connection one form, and has to be included by hand. One of the achievements of our paper is to show that the σ field is indeed coming from a connection one form, but of larger symmetry, and is the one breaking a left-right symmetry.

Indeed in this paper we go one step higher in the construction of the noncommutative manifold, in a sort of noncommutative geometry grand unification. Chamseddine, Connes and Marcolli have shown that the smallest nontrivial almost commutative manifold correspond to the standard model. Here we consider a larger algebra, which we term grand algebra, which corresponds to a different symmetry. It is known that, although the spectral action requires the unification of interactions at a single scale, the usual grand unified theories, such as $SU(5)$ or $SO(10)$, do not fit in the noncommutative geometry framework, and are possible only renouncing to associativity [25, 26]. We point out there is a “next level” in noncommutative geometry, but that it is intertwined with the Riemannian and spin structure of spacetime, and therefore it naturally appears at a high scale. The added degrees of freedom are related to the Riemann-spin structure of spacetime, which can emerge as a symmetry breaking very similar in nature to the Higgs mechanism. This puts in a new light also the phenomenon of fermion doubling [7, 27–29] present in the theory. The remarkable added bonus is that the scalar field necessary for the correct mass of the Higgs naturally appears as the field which drives the breaking of the left-right symmetry.

This paper is organized as follows. In section 2 we introduce briefly the spectral triple constructions. In Sect. 2.4 in particular we describe in detail the Hilbert space and the representation of the algebras. In Sect. 3 we describe the breaking of the symmetries and discuss the predictions for the mass of the Higgs. We also discuss the role of the field σ and in which way it is different from the other bosons. In Sect. 4 we introduce the higher symmetry described by a larger algebra. In Sect. 5 we describe how the breaking of this symmetry puts the σ field on the same footing of the Higgs and the other bosons, as part of the connection one-form. In this section we also show how the breaking of the grand symmetry gives the spin structure of spacetime. A final section contains conclusions and some speculative comments.

2 The spectral triple construction

In this section we introduce some of the mathematical basis for the construction. From a topological point of view, it has been known since the work of Gelfand that a (locally compact) topological space is nothing but a commutative (C^* -)algebra. In a rigorous way, one has that the category of locally compact spaces is (anti)-equivalent to the category of commutative C^* -algebras. The functor passing from one to the other is the one associating to any locally compact space X the commutative algebra $C_0(X)$ of complex functions vanishing at infinity, with inverse the functor associating to any commutative C^* -algebra

\mathcal{A} its pure state space $\mathcal{P}(\mathcal{A})$, that is the extreme points of the set of positive linear map $C_0(X) \rightarrow \mathbb{C}$ with norm 1. One has the isomorphism of algebras,

$$\mathcal{A} \simeq C_0(\mathcal{P}(\mathcal{A})), \quad (2.1)$$

and the homeomorphism of topological spaces

$$X \simeq \mathcal{P}(C_0(X)). \quad (2.2)$$

From a physical point of view, Gelfand theorem simply says that a point x , instead of being viewed as an object which is acted upon by a function f to give a number $f(x)$, can be equivalently seen as an object δ_x that acts on a function to yield a number: $\delta_x(f) := f(x)$. The last point of view is more in the spirit of quantum mechanics, where it is the algebra of observables - not the space - that it is given first.

For physics, topology is not enough. Connes noncommutative geometry is a proposal to extend Gelfand duality between spaces and algebra beyond topology, so that to encompass all the aspect of usual (i.e. Riemannian) differential geometry: in particular the differential structure, the metric aspect, and eventual spin properties.

We first recall how to build a Riemannian (compact) manifold from purely algebraic data (Connes reconstruction theorem), and how to extend this construction to the non-commutative setting. Then we focus on a particular class of such “noncommutative manifolds”, the so called *almost commutative geometries*, which are particularly relevant for the standard model.

2.1 Reconstruction theorem of Riemannian manifolds

The basic device in the construction is a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consisting of

- a commutative $*$ -algebra \mathcal{A} of bounded operators in a Hilbert space \mathcal{H} - containing the identity operator - and a non-necessarily bounded self-adjoint operator D with compact resolvent such that the commutator $[D, a]$ is bounded for all elements in \mathcal{A} ;
 - five conditions which are the translation into an algebraic setting of some precise geometrical properties of a manifold. We list them below, indicating between parenthesis what classical properties they refer to, in case this is not transparent from the name:
1. *Dimension*: there exists an integer $n \in \mathbb{N}$ such that the inverse D^{-1} of the D operator is an infinitesimal of order $\frac{1}{n}$. This means that the characteristic values of D^{-1} goes to 0 as $\frac{1}{n}$. Here D^{-1} means the inverse of the operator D restricted to its domain.

2. *First order condition* (the exterior derivative is a first order differential operator): $[[D, a], b] = 0$ for all $a, b \in \mathcal{A}$.
3. *Smoothness of the coordinates*: for any $a \in \mathcal{A}$, both a and $[D, a]$ belong to the domain of δ^m , for any integer m . Here δ denotes the derivation of \mathcal{A} : $\delta(a) := [[D], a]$.
4. *Orientability* (existence of a chiral structure): this conditions only holds for even dimensional triples. In this case there is a *chirality* operator, usually called γ or Γ , such that $\Gamma^2 = 1$, Γ commutes with \mathcal{A} and anticommutes with D ($\Gamma D = -D\Gamma$). We refer to the literature which shows that this operator is an Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A})$.
5. *Finiteness* (tangent bundle): \mathcal{A} is a pre C^* -algebra and the intersection $\bigcap_{k \in \mathbb{N}} \text{Dom} D^k$ is a finite projective module.

Connes reconstruction theorem [9] then guarantees that:

- i. To any compact Riemannian manifold is associated a canonical spectral triple made of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M , acting on the Hilbert space $\mathcal{H} = \Omega^\bullet(M)$ of square summable differential forms, and $D = d + d^\dagger$ is the Hodge Dirac-operator (d is the exterior derivative, d^\dagger its adjoint).
- ii. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where \mathcal{A} is commutative, then one can construct a smooth Riemannian manifold such that \mathcal{A} is isomorphic to $C^\infty(M)$.

Adding two more conditions,

6. *Poincaré-duality*: the additive coupling on the K -theory of \mathcal{A} determined by the index of the Dirac operator is non degenerate, that is the intersection form $K_*(\mathcal{A}) \times K_*(\mathcal{A}) \rightarrow \mathbb{Z}$ is invertible.
7. *Real structure*: there exists an antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ such that $JaJ^{-1} = a^*$ for all $a \in \mathcal{A}$, and

n mod 8	0	1	2	3	4	5	6	7
$J^2 = \pm i$	+	+	-	-	-	-	+	+
$JD = \pm DJ$	+	-	+	+	+	-	+	+
$J\Gamma = \pm \Gamma J$	+		-		+		-	

one defines a *real spectral triple*. This allows to extend the reconstruction theorem to spin (and spin^c) manifolds.

2.2 Noncommutative manifolds

The seven conditions above are easily extended to the case in which \mathcal{A} is noncommutative: conditions 1, 3 and 5 remain unchanged. As a consequence of the noncommutativity of the algebra, one asks for an action of the opposite algebra \mathcal{A}° (identical to \mathcal{A} as a vector space, but with reversed product: $a^\circ b^\circ = (ba)^\circ$) that commutes with the action of \mathcal{A} . This is achieved by identifying b° with Jb^*J^{-1} , i.e. by requiring one more condition on the real structure J :

$$7'. [a, b^\circ] = 0 \text{ for all } a, b \in \mathcal{A}.$$

The remaining conditions are modified accordingly

$$2'. [[D, a], b^\circ] = 0 \text{ for all } a, b \in \mathcal{A}.$$

$$4'. \text{ The Hochschild cycle is in } Z_N(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ).$$

$$6'. \text{ The cup product by } \mu \in KR^n(\mathcal{A} \otimes \mathcal{A}^\circ) \text{ is an isomorphism.}$$

For more details, in particular on the mathematical tools involved in Poincaré condition, the reader is invited to see the original paper [30]. In this work the two conditions that are of particular importance are the grading condition $[\Gamma, a] = 0$, and the first order condition 2'.

A spectral triple with noncommutative algebra \mathcal{A} , satisfying the 7 conditions above is a natural candidate as a *noncommutative manifold*.

2.3 Almost Commutative Manifold and the Standard Model

A particular form of noncommutative manifolds are the *almost commutative* ones. In this case the noncommutativity is given by the tensor product of a commutative algebra (continuous functions on an ordinary manifold) times a finite dimensional noncommutative algebra. There is a spacetime (the commutative manifold), enriched at every point by a noncommutative algebra, which gives an internal structure, on which the gauge group lives. The reconstruction theorem [9] shows that if the algebra \mathcal{A} in $(\mathcal{A}, \mathcal{H}, D)$ is commutative, then the spectral triple is of the form $(C^\infty(M), L^2(M, S), \not{D})$, canonically associated to a Riemannian spin manifold M . Here $S \rightarrow M$ is a spinor bundle and \not{D} is the corresponding Dirac operator. In this framework a gauge theory, with group of invariance G , is fully geometrized and it is on the same footing as gravity. The former, in fact, emerges as the gauge theory of the inner automorphisms of the algebra

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F \tag{2.3}$$

where \mathcal{A}_F is the smallest $*$ -algebra containing G as the group of its unitary elements. Analogously, the latter can be seen as the gauge theory of diffeomorphisms of M , which

are nothing but the outer automorphisms of \mathcal{A} . As far as the Hilbert space \mathcal{H} is concerned a suitable choice is to take

$$\mathcal{H} = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F \quad (2.4)$$

where $sp(L^2(\mathbb{R}^4))$ are square summable spinors and $\mathcal{H}_F = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c$ contains all the 96 degrees of freedom: 8 fermions (electrons, neutrinos, up and down quarks with three colours each), for two chiralities for N=3 families plus antiparticles.

The chiral and real structure are $\Gamma = \gamma^5 \otimes \gamma_F$ and $\mathcal{J} = J \otimes J_F$ where the finite part are given by

$$\gamma_F = \begin{pmatrix} -\mathbb{I}_{8N} & & & \\ & \mathbb{I}_{8N} & & \\ & & -\mathbb{I}_{8N} & \\ & & & \mathbb{I}_{8N} \end{pmatrix}, \quad J_F = \begin{pmatrix} 0 & \mathbb{I}_{16N} \\ \mathbb{I}_{16N} & 0 \end{pmatrix} cc \quad (2.5)$$

with cc usual complex conjugation. γ^5 is the usual chirality defined in terms of the four gamma matrices and J is the charge conjugation operator.

The operator D for this spectral triple is a generalization of \not{D} , and we will call it still the Dirac operator of the almost commutative geometry. It contains all information relative to the masses of particles, i.e. the Yukawa couplings. The fermionic action is:

$$S_F = \overline{J\psi}(D + A + JAJ)\psi \quad (2.6)$$

and in this way the part containing D gives masses to the fermions, and the potentials are correctly coupled.

The operator D can be explicitly written as

$$D = \not{D} \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F \quad (2.7)$$

where

$$D_F = \begin{pmatrix} 0 & \mathcal{M} & \mathcal{M}_R & 0 \\ \mathcal{M}^\dagger & 0 & 0 & 0 \\ \mathcal{M}_R^\dagger & 0 & 0 & \mathcal{M}^* \\ 0 & 0 & \mathcal{M}^T & 0 \end{pmatrix} \quad (2.8)$$

The matrix \mathcal{M} contains the Dirac masses, or rather in the renormalized theory, the Yukawa couplings, while \mathcal{M}_R contains Majorana masses:

$$\mathcal{M} = \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} \quad (2.9)$$

$$\mathcal{M}_R = \begin{pmatrix} M_R & 0 \\ 0 & 0 \end{pmatrix} \quad (2.10)$$

where M_u is a diagonal matrix containing the masses of the up, charm and top quarks and the neutrino (Dirac mass), M_R contains the Majorana neutrino masses and M_d is

a diagonal matrix containing the remaining quarks and electrons, muon and tau masses multiplied by the Cabibbo-Kobayashi-Maskawa mixing matrix.

In all these considerations the Dirac operator is a datum of the problem, i.e. the fermion masses, or better the Yukawa couplings, are known quantities, as well as the structure of D_F , which is “sparse” matrix, meaning than most of its entries are zeros. Supersymmetry may be described “filling” some of these voids with the bosonic superpartners of the fermions [31].

Notice that since the Hilbert space defined in (2.4) is the tensor product of four dimensional spinors by the 96-dimensional elements of \mathcal{H}_F , the elements of the Hilbert space have in reality 384 dimension, 128 for a single generation. This redundancy of states is known as fermion doubling [27–29]. The problem is not only the overcounting, but the presence of states which do not have a definite parity, being the product of a right and left handed state. These states have to be projected out only in the fermionic sector. This can be done considering the fermionic determinant to be a Pfaffian [7], while at the same time the D operator should act, as far the bosonic spectral action is concerned, on the full \mathcal{H} . We will see in the following that this doubling may be an essential feature of the model.

This Dirac operator reproduces the classical fermionic action with fermion masses. The bosonic part of the action, and more importantly the quantization of the fields can be achieved via the spectral action principle or via considerations related to spectral regularization and the role of anomalies.

Rather than isomorphisms of algebras, a natural notion of equivalence for noncommutative (C^*) -algebras is Morita equivalence [32]. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an algebra \mathcal{B} that is Morita equivalent to \mathcal{A} , one can define a spectral triple $(\mathcal{B}, \mathcal{H}', D')$ on \mathcal{B} . Interestingly, upon taking \mathcal{B} to be \mathcal{A} , this leads to a whole family of spectral triples $(\mathcal{A}, \mathcal{H}, D_A)$ where $D_A \equiv D + A$ with $A \in \Omega_D^1(\mathcal{A})$ self-adjoint with

$$\Omega_D^1(\mathcal{A}) \equiv \left\{ \sum_i a_i [D, b_i] : a_i, b_i \in \mathcal{A} \right\}. \quad (2.11)$$

The bounded operators A are generally referred to as the inner fluctuations of D and are interpreted as gauge fields. From the point of view of noncommutative geometry the gauge fields are nothing but fluctuations of the internal geometry.

When considering a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$, we have the additional restriction that the real structure J' of the spectral triple $(\mathcal{A}, \mathcal{H}', D'; J')$ on the Morita equivalent algebra should be compatible with the relation coming from condition 7' above (for an even number of dimensions): $J'D' = D'J'$. Upon taking \mathcal{B} to be \mathcal{A} again in such a case, the resulting spectral triple is of the form $(\mathcal{A}, \mathcal{H}, D_A; J)$, but now with

$$D_A \equiv D + A + JAJ^*. \quad (2.12)$$

Note that in the commutative case these inner fluctuations vanish. The gauge group is defined to be the unitary elements of the algebra: $U(\mathcal{A}) \equiv \{u \in \mathcal{A} : uu^* = u^*u = 1\}$. It

acts on elements ψ in the Hilbert space via $\psi \rightarrow \psi^u = uJuJ^*\psi$. This induces an action on D_A as $D_A \rightarrow uD_Au^*$. Consequently, the inner fluctuations transform as

$$A \longrightarrow A^u = uAu^* + u[D, u^*]. \quad (2.13)$$

In this framework both the Higgs field and the vector bosons mediating the three fundamental interactions emerge spontaneously as “internal or external” connections, 1-forms combinations of the elements of the algebra with the appropriate Dirac operator.

2.4 Algebras and Representations on the Hilbert Space

We now look for the almost commutative manifold which describes a viable gauge theory. The remarkable feature is that, among all possible Yang-Mills models, the standard model is singled out. Let us very briefly sketch the relevant aspects of noncommutative geometry construction leading to the standard model [7] (for a review see also [8]). The most general finite algebra that satisfies all conditions for the noncommutative space to be a manifold is [7]

$$\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \quad (2.14)$$

This algebra acts on an Hilbert space of dimension $2(2a)^2$, as shown in [7].

To have a non-trivial grading on $\mathbb{M}_a(\mathbb{H})$ the integer a must be at least 2. Thus the simplest possibility is

$$\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C}) \quad (2.15)$$

The grading condition $[a, \Gamma] = 0$ reduces immediately the algebra to the left-right algebra:

$$\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}) \quad (2.16)$$

This is basically a Pati-Salam model [33], one of the not many models allowed by the spectral action [34]. The order one condition reduces further the algebra to the one of the standard model; i.e. the algebra whose unimodular group is $U(1) \times SU(2) \times U(3)$. An element of this algebra is a combination of a complex number, a quaternion and a three by three matrix:

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \quad (2.17)$$

We will always represent quaternions as 2×2 matrices of the form: $\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ where $x, y \in \mathbb{C}$ and \bar{x} means complex conjugation:

We need to represent these algebra on the Hilbert space (2.4), its elements are 384 components vectors. The number 384 comes from degrees of freedom which have different physical meaning. Some of them refer to “internal” degrees of freedom, like colour, some refer to the Riemannian-spin structure, and have a spacetime meaning. Let us give a complete and detailed description of all degrees of freedom:

$$\Psi_{a\dot{a}\alpha}^{\text{IC}m}(x) \in \mathcal{H} = L^2(\mathbb{R}_4) \otimes \mathbf{H}_F = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F \quad (2.18)$$

Note the difference between \mathbf{H}_F and \mathcal{H}_F . The latter is the 96 dimensional space (2.4) used normally in the literature, and its spinors are to be multiplied by *spinors*, while the former is the larger 384 dimensional space which exhibits explicitly the fermion doubling overcounting. So far the Hilbert space has been considered always in its factorized form involving \mathcal{H}_F . One of the novelties of this work is the fact that we will consider algebras which do not act in separately on spinors and the internal part.

The extra degrees of freedom have to be projected out to avoid states which are left chiral in the inner indices and right chiral in the outer ones, or viceversa. Since the total chirality Γ is the product of γ_F which acts in the inner indices of \mathcal{H}_F times γ_5 which acts on the spin indices, the spurious states are the ones for which $\Gamma\Psi = -\Psi$. Taking the functional integral of the fermionic action to be a Pfaffian [7] takes care of the extra factors, but one cannot simply project out the extra states and work with a representation on a smaller Hilbert space in the bosonic spectral action, since in order to obtain the proper action of the standard model coupled with gravity all degrees of freedom are necessary [27].

The meaning of the various indices of $\Psi_{a\dot{a}\alpha}^{\text{IC}m}(x)$ is the following:

$a = l, r$
 $\dot{a} = \dot{l}, \dot{r}$ These are the spinor indices. They take two values each, and together they make the four indices on an ordinary Dirac spinor. The index $a = l, r$ indicates chirality and runs over the left right part of the spinor, while the dotted index differentiates particles from antiparticles. Note that with our conventions a left antiparticle actually has right handed chirality. Written in the usual chiral basis these degrees of freedom are

$$\xi = \begin{pmatrix} \xi_{l\dot{l}} \\ \xi_{r\dot{r}} \\ \xi_{r\dot{l}} \\ \xi_{l\dot{r}} \end{pmatrix} \quad (2.19)$$

$\alpha = 1 \dots 4$ The flavour index. It runs over the set u_R, d_R, u_L, d_L when $I = 1, 2, 3$, and ν_R, e_R, ν_L, e_L when $I = 0$. And it repeats in the obvious way for the other generations.

$I = 0, \dots 3$ Indicates a “lepto-colour” index. The zeroth “colour” actually identifies leptons while $I = 1, 2, 3$ are the usual three colours of QCD.

$C = 0, 1$ This indicates whether we are considering “particles” ($C = 0$) or “antiparticles” ($C = 1$). We will see that until the division between internal space and spacetime is made the distinction between particles and antiparticles does not necessarily correspond with the usual one.

$m = 1, 2, 3$ The generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters which appears in it.

For the remainder of this paper the generation index m does not play any role. We will therefore usually suppress it and work with one generation, thus effectively considering \mathcal{H}_F to have 32 dimensions, and \mathbf{H}_F to be 128-dimensional.

The operator J basically exchanges the two blocks corresponding to particles and antiparticles both in \mathcal{H}_F and on the spinor indices and acts with complex conjugation cc :

$$(J\Psi)_{a\dot{a}\alpha}^{\text{IC}}(x) = \varepsilon_{\text{CD}}\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}{}_{cc}\Psi_{b\dot{b}\alpha}^{\text{ID}}(x) \quad (2.20)$$

where ε is the antisymmetric tensor in two indices. Written as a tensor J is

$$J = \delta_{\text{IJ}}\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\delta^{\alpha\beta}\varepsilon_{\text{CD}}cc \quad (2.21)$$

The chirality acts as

$$(\Gamma\Psi)_{a\alpha}^{\text{Ic}} = \gamma_5^{ab}\epsilon^\alpha\Psi_{b\alpha}^{\text{Ic}} \quad (2.22)$$

where $\epsilon_\alpha = 1$ if $\alpha = 1, 2$ i.e. if the index represents u_L, d_L, ν_L, e_L , $\epsilon_\alpha = -1$ otherwise, γ_5 is the usual product of the four γ matrices in four dimension, in the double index notation. Spurious states with the “wrong” chirality are present, due to the fermion doubling.

We now give the representations of the finite dimensional algebras on \mathbf{H}_F . Let us first consider \mathcal{A}_F defined in (2.15). Consider a generic element $A = \{Q, M\}$ with $Q \in M_2(\mathbb{H})$ and $M \in M_4(\mathbb{C})$, then, writing explicitly all indices we have:

$$A_{\text{IJCD}}^{ab\dot{a}\dot{b}\alpha\beta} = \delta^{ab}\delta^{\dot{a}\dot{b}}(\delta_{\text{C0}}\delta_{\text{IJ}}Q^{\alpha\beta} + \delta_{\text{C1}}\delta^{\alpha\beta}M_{\text{IJ}}) \quad (2.23)$$

where we have considered the quaternion matrix $M_2(\mathbb{H})$ as a 4×4 matrix. Note that the two Kronecker δ at the beginning of the expression of A show that the algebra acts in a diagonal way on the spin indices. The rest of the index structure is such that, exchanging the particle and the antiparticle blocks, i.e. considering $J\mathcal{A}_F J$, the elements of the matrices commute. In this case there is no more room for the representation of a larger algebra if one wants to keep the commutativity, unless more fermions are added.

We now give the representation of \mathcal{A}_{sm} , we will indicate its elements as $a = \{\mathbf{m}, \mathbf{q}, \mathbf{c}\}$ with $\mathbf{m} \in M_3(\mathbb{C})$, $\mathbf{q} \in \mathbb{H}$, $\mathbf{c} \in \mathbb{C}$. We have:

$$a_{\text{IJCD}}^{ab\dot{a}\dot{b}\alpha\beta} = \delta^{ab}\delta^{\dot{a}\dot{b}}(\delta_{\text{C0}}\delta_{\text{IJ}}(\tilde{\mathbf{q}}^{\alpha\beta} + \tilde{\mathbf{c}}^{\alpha\beta}) + \delta_{\text{C1}}\delta^{\alpha\beta}(\hat{\mathbf{m}}_{\text{IJ}} + \hat{\mathbf{c}}_{\text{IJ}})) \quad (2.24)$$

where we have used the following 4×4 matrices:

$$\begin{aligned} \hat{\mathbf{m}} &= \begin{pmatrix} 1 & & & \\ & \mathbf{m} & & \end{pmatrix} \\ \tilde{\mathbf{q}} &= \begin{pmatrix} \mathbb{I}_2 & & & \\ & \mathbf{q} & & \end{pmatrix} \\ \hat{\mathbf{c}} &= \begin{pmatrix} \mathbf{c} & & & \\ & \mathbb{I}_3 & & \end{pmatrix} \\ \tilde{\mathbf{c}} &= \begin{pmatrix} \mathbf{c} & & & \\ & \bar{\mathbf{c}} & & \\ & & \mathbb{I}_2 & \end{pmatrix} \end{aligned} \quad (2.25)$$

Again Kronecker δ 's at the beginning of the l.h.s. in (2.24) indicate that the algebra does not act on the spinor indices. The two terms in the parenthesis indicate the different action of the algebra. It appears that the action on particles and antiparticles is different, but the fact that the fluctuations of the Dirac operator in (2.12) involve J in an essential way reinstates the symmetry of the action. Note also that the representation is built in such a way that the quaternions act only on the left particles, while the elements of M_3 act only on quarks.

The gamma matrices act only on the a and \dot{a} indices, while the algebra \mathcal{A}_{sm} acts on the inner indices. This means that the Dirac operator is factorized as well, as it happens in (2.7), the term \not{D} containing the γ 's acting only on the spinor indices, and a 32×32 matrix which acts on the remaining indices.

3 Symmetry breaking and the mass of the Higgs

In the scheme that we are describing here the Higgs appears on the same footing as the other (vector) boson as part of the connection one form [5, 35]. The remarkable fact is that its mass is not independent on the Yukawa couplings of the fermions, and therefore the theory has important predictive power. While it is already an important achievement that the predicted mass results of the correct order of magnitude, lately the theory is reaching the point for which its predictions start to be of phenomenological interest. In this section we recall how the calculation of the Higgs is performed, and the role of the new scalar field σ . Those familiar with this calculation can skip this section,

3.1 Reduction of the \mathcal{A}_F algebra

In this subsection we describe how the requirement that the algebra commutes with the chirality reduces $\mathcal{A}_F \rightarrow \mathcal{A}_{LR}$ and how the order one condition further reduces $\mathcal{A}_{LR} \rightarrow \mathcal{A}_{sm}$. This is not a dynamical or spontaneous breaking, as it is forced by the mathematical requirement of the order one condition. Therefore there was never a phase in which the algebra was the full \mathcal{A}_F .

It is easy to see that the requirement of the algebra commuting with Γ requires the elements of $M_2(\mathbb{H})$ to be block diagonal and reduces this part of the algebra to $\mathbb{H}_L \oplus \mathbb{H}_R$, i.e. we have $\mathcal{A}_F \rightarrow \mathcal{A}_{LR}$. This means that $Q_{\alpha\beta}$ in (2.23) acts separately on the left and right handed doublets. At this point the gauge would therefore be (taking unimodularity into account) $SU(2)_L \times SU(2)_R \times SU(4)$. As shown in [7] the further breaking to the standard model group is a consequence of the presence of right handed neutrinos. We will recall it in the next section.

The order one condition is,

$$[[D_F, a], b^\circ] = 0 ; \quad a, b \in \mathcal{A}_{LR} \quad (3.1)$$

Using the tensorial notations of [10] we write

$$a = \begin{pmatrix} \delta^{IJ} X_{\alpha\beta} & 0 \\ 0 & Y^{IJ} \delta_{\alpha\beta} \end{pmatrix} \quad (3.2)$$

$$b^\circ = JbJ = \begin{pmatrix} W^{IJ} \delta_{\alpha\beta} & 0 \\ 0 & \delta^{IJ} Z_{\alpha\beta} \end{pmatrix}. \quad (3.3)$$

Assuming first there is no Majorana mass term, the Dirac operator is block diagonal [10], $D_F = \text{diag}(D_{AB}, \bar{D}_{AB})$, where D_{AB} has components

$$D_{\alpha\beta}^{IJ} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & k_\nu^* & 0 \\ 0 & 0 & 0 & k_e^* \\ k_\nu & 0 & 0 & 0 \\ 0 & k_e & 0 & 0 \end{pmatrix} & 0 \\ 0 & \mathbb{I}_3 \begin{pmatrix} 0 & 0 & k_u^* & 0 \\ 0 & 0 & 0 & k_d^* \\ k_u & 0 & 0 & 0 \\ 0 & k_d & 0 & 0 \end{pmatrix} \end{pmatrix}. \quad (3.4)$$

The order 1 condition thus becomes

$$[[D_F, a], b^\circ] = \begin{pmatrix} [[D_{AB}, X], W] & 0 \\ 0 & [[\bar{D}_{AB}, Y], Z] = 0 \end{pmatrix} \quad (3.5)$$

This can be made explicit by writing:

$$(D_{\alpha\gamma}^{IK} X_{\gamma\beta} - X_{\alpha\gamma} D_{\gamma\beta}^{IK}) W^{KJ} - W^{IK} (D_{\alpha\gamma}^{KJ} X_{\gamma\beta} - X_{\alpha\gamma} D_{\gamma\beta}^{KJ}) = 0 \quad (3.6)$$

In the matrix form, defining

$$X_{\alpha\beta} = \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ -a_2^* & a_1^* & 0 & 0 \\ 0 & 0 & a_3 & a_4 \\ 0 & 0 & -a_4^* & a_3^* \end{pmatrix}; \quad W^{IJ} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}$$

one finds (e.g. using symbolic manipulation programs) that a generic term of (3.6) is of kind

$$\begin{aligned} (k_u a_1 - a_3 k_u) f_{j1} - f_{j1} (k_\nu a_1 - a_3 k_\nu) &= 0, \quad j = 2, 3, 4 \\ (k_u a_2 - a_4 k_d) f_{j1} - f_{j1} (k_\nu a_2 - a_4 k_\nu) &= 0, \quad j = 2, 3, 4 \end{aligned} \quad (3.7)$$

and similar equations for f_{1j} . Apart from the trivial solution the relations are satisfied imposing $f_{j1} = f_{1j} = 0$.

At this stage, without Majorana coupling in the Dirac operator we have the reduction $\mathbb{M}_4(\mathbb{C}) \rightarrow \mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})$. The introduction of a Majorana coupling k_R (at this point a

constant) in the Dirac operator induces new terms in (3.5), that in explicit form leads to new constraints:

$$\begin{aligned}(a_1 k_R - f_{11} k_R) b_2^* &= 0 \\ a_2 k_R b_2^* &= 0\end{aligned}\tag{3.8}$$

leading to $a_1 = f_{11}$, and $a_2 = 0$ that means $(\mathbb{H} \oplus \mathbb{H}) \oplus (\mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})) \rightarrow (\mathbb{C}' \oplus \mathbb{H}) \oplus (\mathbb{C} \oplus \mathbb{M}_3(\mathbb{C}))$ with $\mathbb{C} = \mathbb{C}'$ and so the “Standard Model algebra” $\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})$.

3.2 The spectral action

In this section we recall only the features of the spectral action, referring to the original articles for the full treatment. The spectral action is based on a regularization of the spectrum of the fluctuated Dirac operator D_A defined in (2.12): it reads:

$$S_B = \text{Tr} \chi \left(\frac{D_A^2}{\Lambda^2} \right)\tag{3.9}$$

where χ is a cutoff function, usually the (smoothened) characteristic function on the interval $[0, 1]$, and Λ is a renormalization scale. The spectral action has an expansion in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n a_n (D^2/\Lambda^2)\tag{3.10}$$

where the f_n are the momenta of χ

$$\begin{aligned}f_0 &= \int_0^\infty dx x \chi(x) \\ f_2 &= \int_0^\infty dx \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \Big|_{x=0} \quad n \geq 0\end{aligned}\tag{3.11}$$

and the a_n are the Seeley-de Witt coefficients [36, 37], which are nonzero only for n even. Considering D_A^2 of the form

$$D^2 = g^{\mu\nu} \partial_\mu \partial_\nu + \alpha^\mu \partial_\mu + \beta\tag{3.12}$$

define

$$\begin{aligned}\omega_\mu &= \frac{1}{2} g_{\mu\nu} (\alpha^\nu + g^{\sigma\rho} \Gamma_{\sigma\rho}^\nu) \\ \Omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \\ E &= \beta - g^{\mu\nu} (\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma_{\mu\nu}^\rho \omega_\rho)\end{aligned}\tag{3.13}$$

the form of the first three coefficients is:

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2} \int d^4x \sqrt{g} \operatorname{tr}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2} \int d^4x \sqrt{g} \operatorname{Tr} \left(-\frac{R}{6} + E \right) \\
a_4 &= \frac{1}{16\pi^2} \frac{1}{360} \int d^4x \sqrt{g} \operatorname{Tr} (-12\nabla^\mu \nabla_\mu R + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \\
&\quad + 2R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu \nabla_\mu E + 30\Omega_{\mu\nu} \Omega^{\mu\nu}) \quad (3.14)
\end{aligned}$$

When applying this to the Dirac operator (2.7) one finds in the expansion the Lagrangian of the standard model coupled with gravity [7, Sect. 4.1]. However the parameters related to the Higgs come out to be functions of the other parameters in D_F , that is the Yukawa couplings, which are in turn dominated by the top quark coupling. In this sense the model predicts the Higgs mass as a function of the other gauge couplings, the Yukawa top mass and the scale Λ which appears also as the scale in which the three gauge couplings coincide.

3.3 Mass of the Higgs

In the Standard Model the Higgs mass is fixed by the vacuum expectation value of the Higgs field and by the autointeraction coupling constant $\lambda(t)$ evaluated at the low-energy scale M_Z ,

$$m_H = \sqrt{2\lambda_0} 246 \text{ GeV} \quad (3.15)$$

where 246 GeV is the v.e.v. Without additional scalar field the running coupling constant for the autointeraction coupling λ is,

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2} \left(24\lambda^2 - (3g_1^2 + 9g_2^2 + 12y_{top}^2) \lambda + \frac{6}{16} (g_1^4 + 2g_1^2 g_2^2 + 3g_2^4) - 3y_{top}^4 \right) \quad (3.16)$$

and the running coupling constant for g_1 , g_2 , and y_{top} are well known:

$$\frac{dy_{top}}{dt} = \frac{1}{16\pi^2} \left(\frac{9}{2} y_{top}^2 - \frac{17}{12} g_1^2 - \frac{9}{4} g_2^2 - 8g_3^2 \right) \quad (3.17)$$

$$\frac{dg_i}{dt} = \frac{1}{16\pi^2} b_i g_i^3 \quad \text{with } b_i = \left(\frac{41}{6}, -\frac{19}{6}, -7 \right) \quad (3.18)$$

these equations are solved with the following initial condition dictated by experimental values:

$$\begin{aligned}
g_1(M_Z) &= 0.3575 \\
g_2(M_Z) &= 0.6514 \\
g_3(M_Z) &= 1.221 \\
y_{top}(M_Z) &= 0.993
\end{aligned} \quad (3.19)$$

The spectral action requires an unique unification scale, which as is known is true only in an approximate sense. If one takes the unification scale to be $\Lambda = 10^{17}\text{GeV}$ then the bosonic action gives [7, Eq. 5.6],

$$\lambda(\Lambda) = \frac{4}{3}g_3(\Lambda)^2 \sim 0.356 \quad (3.20)$$

The numerical solution to these equations gives $\lambda(M_Z) \sim 0.241$ and a Higgs mass of the order of 170 GeV. This value is not in agreement with the recent LHC experiments [38].

One can think of extending the model to solve this. There have been several expansions of this kind of models, and some of them are reviewed in [6]. In particular C. Stephan has proposed in [39] that the presence of an extra scalar field, corresponding to the breaking of a extra U(1) symmetry, can bring down the mass of the Higgs to the value of 126 GeV. This model however does contain extra fermions. Earlier examples of extensions can be found in [40–46].

Recently, in [24] the noncommutative geometry model was enhanced to also overcome the high energies instability of a Higgs boson with mass around 126 GeV, in addition to predicting the correct mass. This is done ruling out the hypothesis of the “big desert” and considering an additional scalar field that lives at high energies and that gives mass to the Majorana neutrinos.

A good prediction for the Higgs mass could be obtained considering the Higgs field coupled to a new σ scalar field,

$$V(H, \sigma) = \frac{1}{4} (\lambda_H H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2 \sigma^2) \quad (3.21)$$

where λ_H , λ_σ , $\lambda_{H\sigma}$ are defined in the spectral action and hold to the unification scale,

$$\begin{aligned} \lambda_H &\equiv \frac{\rho^4 + 3}{(3 + \rho^2)^2} 4g^2 \\ \lambda_{H\sigma} &\equiv \frac{2\rho^2}{\rho^2 + 3} 4g^2 \\ \lambda_\sigma &\equiv 8g^2 \end{aligned} \quad (3.22)$$

with $g = g_3(\Lambda)$ and $\rho = y_\nu/y_{top}$ is a free parameter of the theory.

Writing the RG equations for the top quark, neutrino, Higgs and singlet quartic cou-

plings we have [47, 48],

$$\begin{aligned}
\frac{dy_{top}}{dt} &= \frac{y_{top}}{16\pi^2} \left(\frac{9}{2}y_{top}^2 + y_\nu^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \right) \\
\frac{dy_\nu}{dt} &= \frac{y_\nu}{16\pi^2} \left(3y_{top}^2 + \frac{5}{2}y_\nu^2 - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2 \right) \\
\frac{d\lambda_{H\sigma}}{dt} &= \frac{\lambda_{H\sigma}}{16\pi^2} \left(6y_{top}^2 + 2y_\nu^2 - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2 + 12\lambda_H + 6\lambda_\sigma + 8\lambda_{H\sigma} \right) \\
\frac{d\lambda_\sigma}{dt} &= \frac{1}{16\pi^2} (8\lambda_{H\sigma}^2 + 18\lambda_\sigma^2) \\
\frac{d\lambda_H}{dt} &= \frac{1}{16\pi^2} (24\lambda_H^2 - (3g_1^2 + 9g_2^2) \lambda_H + 2\lambda_{H\sigma}^2 \\
&\quad + \frac{6}{16} (g_1^4 + 2g_1^2g_2^2 + 3g_2^4) + (12y_{top}^2 + 4y_\nu^2) \lambda - (3y_{top}^4 + y_\nu^4)) \quad (3.23)
\end{aligned}$$

These equations are solved by using the boundary conditions 3.19, 3.22 with a particular choice of ρ . The running of the Higgs mass is given by a new relation due to the presence of the new scalar field,

$$m_H(0) = 246 \sqrt{2\lambda_H(0) \left(1 - \frac{\lambda_{H\sigma}^2(0)}{\lambda_H(0)\lambda_\sigma(0)} \right)} \quad (3.24)$$

The result of this procedure lowers the Higgs mass prediction to $m_H \sim 126\text{GeV}$.

The origin of the field is however quite different from the Higgs field. The latter, as well as all the other bosons, come from the commutator of the finite dimensional part of the Dirac operator, D_F defined in (2.8) with elements of the algebra. The matrix D_F is composed of numbers, constants without space dependence. When these numbers are commuted with elements of the algebra they are promoted to fields. They cannot all be fields to start with, otherwise the model would lose its predictive power, in that all Yukawa couplings would be promoted to fields, and the masses of all fermions would run independently, thus making any prediction based on the Higgs impossible. One of the entries of D_F corresponds to the Majorana mass for neutrinos. One could think that this entry, considered a Yukawa coupling, would give the required field, in the right place. The problem however is that in taking the commutator with elements of the algebra this term does not contribute to the one-form, and hence to the potential. This forced the authors of [24] to “promote” only the entry k_R corresponding to the Majorana mass to a field, in a somewhat arbitrary way.

In the following we will see that there is indeed a field in this position if we consider a larger symmetry.

4 The Grand Algebra

In this section we show that it is possible to go to a higher symmetry in this framework. This larger symmetry is however very distinct from the usual grand unified theory, where all gauge groups are actually subgroups of a grand unified group, such as $SU(5)$ or $SO(10)$. A naive reposition of this scheme is impossible because in the noncommutative geometry framework the gauge group is the unitary part of an algebra, and it inherits its representation. Algebras have less irreducible representation than groups, and finite dimensional noncommutative C^* -algebras have only the fundamental representation and the trivial one. Fortunately in the standard model there appear only colour triplets and singlets, and weak isospin doublets and singlets, thus enabling the construction. Moreover, as we saw in section 2.3, the algebras allowed are of the kind shown in (2.14): $M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})$. We have seen how the smallest nontrivial case $a = 2$ gives rise to \mathcal{A}_F , and then to \mathcal{A}_{sm} . We now present an analysis of the case $a = 4$, and show that its presence enables us to solve the problem of the presence of the extra field σ introduced in section 3.3, and at the same time shows an intimate relationship between the spin structure and the reduction of the algebra, which happens with a mechanism similar to the Higgs.

We will call the algebra

$$\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \quad (4.1)$$

the “grand” algebra. Its representation on \mathcal{H} is more involved than the previous cases. We stress that the Hilbert space we use is the one we have used so far. No new particles are introduced.

The representations of the algebras that we used so far have been diagonal in the spin index, as signified by the presence of the term $\delta^{ab}\delta^{\dot{a}\dot{b}}$ in equations (2.23) and (2.24). By contrast the representation of \mathcal{A}_G mixes the internal and spin indices in a nontrivial way. In analogy with what done earlier we will consider the algebra as two sets of 8×8 matrices, and we will consider both the matrices as having a block structure of four 4×4 matrices blocks. We therefore have the matrices $Q^{\alpha\beta}_{ab}$ and $M^{IJ}_{\dot{a}\dot{b}}$. The representation of the element $A = (Q, M) \in \mathcal{A}_G$ algebra therefore is:

$$A_{CD,ab,\dot{a}\dot{b}}^{IJ,\alpha\beta,mn} = \delta^{mn} (\delta_{C0}\delta^{IJ}\delta_{\dot{a}\dot{b}}Q^{\alpha\beta}_{ab} + \delta_{C1}M^{IJ}_{\dot{a}\dot{b}}\delta^{\alpha\beta}\delta_{ab}) \quad (4.2)$$

The action of these matrices on the spinor is as follows:

$$\begin{aligned} Q : \begin{pmatrix} \mathbb{I}_2^{\dot{a}\dot{b}} \otimes Q_{r,r}^{\alpha\beta} & \mathbb{I}_2^{\dot{a}\dot{b}} \otimes Q_{r,l}^{\alpha\beta} \\ \mathbb{I}_2^{\dot{a}\dot{b}} \otimes Q_{l,r}^{\alpha\beta} & \mathbb{I}_2^{\dot{a}\dot{b}} \otimes Q_{l,l}^{\alpha\beta} \end{pmatrix} &\mapsto \begin{pmatrix} \psi_{l,\dot{a},C}^{\alpha,I,m} \\ \psi_{r,\dot{a},C}^{\alpha,I,m} \end{pmatrix} \\ M : \begin{pmatrix} M_{11}^{IJ} \otimes \mathbb{I}_2^{ab} & M_{12}^{IJ} \otimes \mathbb{I}_2^{ab} \\ M_{21}^{IJ} \otimes \mathbb{I}_2^{ab} & M_{22}^{IJ} \otimes \mathbb{I}_2^{ab} \end{pmatrix} &\mapsto \begin{pmatrix} \psi_{a,1,C}^{\alpha,I,m} \\ \psi_{a,2,C}^{\alpha,I,m} \end{pmatrix} \end{aligned} \quad (4.3)$$

As usual the quaternionic part acts on particles and the complex part on antiparticles. The novelty here is that the quaternionic part acts on the left-right indices (and the complex part on the particle-antiparticle indices) of the spacetime spinors as well, while in the previous cases the action of the finite dimensional algebra was diagonal in these indices. We see that at the grand algebra level the fermion doubling becomes a *necessity*, rather than a nuisance.

The commutation of an element of the algebra \mathcal{A}_G with an element of the opposite algebra $J\mathcal{A}_GJ$ vanishes, this is the order zero condition:

$$[A, JBJ] = 0, \forall A, B \in \mathcal{A}_G. \quad (4.4)$$

This is easily seen noticing that all J does is to exchange the δ_{C0} and δ_{C1} terms in equations (2.23), (2.24) and (4.2), and since these terms act always on different indices the commutation is assured. Without the enlargement the action of the finite dimensional algebra to the spinorial indices it is impossible to find a representation which satisfies the order zero condition. In this respect the grand algebra is not anymore an internal algebra.

In a way very similar to the reduction $\mathcal{A}_F \rightarrow \mathcal{A}_{LR}$, the grading condition reduces $\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ to

$$\mathcal{A}'_G = [M_2(\mathbb{H}) \oplus M'_2(\mathbb{H})] \oplus [M_4(\mathbb{C}) \oplus M'_4(\mathbb{C})] \quad (4.5)$$

To see this we can put Γ in the form,

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{bmatrix} \mathbb{I}_4 \otimes \Gamma_8 & 0 \\ 0 & \mathbb{I}_4 \otimes -\Gamma_8 \end{bmatrix} \quad (4.6)$$

with $\Gamma_8 = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}$. We see that on both sectors, $M_4(\mathbb{H})$ and $M_8(\mathbb{C})$, Γ acts with the matrix $\text{diag}[\mathbb{I}_4 \otimes \Gamma_8, \mathbb{I}_4 \otimes (-\Gamma_8)]$. On the first sector $\text{diag}[\mathbb{I}_4 \otimes \Gamma_8, \mathbb{I}_4 \otimes (-\Gamma_8)]$ must commute with a generic element $a \in \mathbb{I}_8 \otimes M_4(\mathbb{H})$ that we write,

$$A = \mathbb{I}_8 \otimes \begin{pmatrix} Q_1^{\alpha\beta} & Q_3^{\alpha\beta} \\ Q_4^{\alpha\beta} & Q_2^{\alpha\beta} \end{pmatrix}, \quad Q_i^{\alpha\beta} \in M_2(\mathbb{H}), \text{ for } i = 1, 2, 3, 4. \quad (4.7)$$

So the matrix $\begin{pmatrix} Q_1^{\alpha\beta} & Q_3^{\alpha\beta} \\ Q_4^{\alpha\beta} & Q_2^{\alpha\beta} \end{pmatrix}$ has to commute with the matrix Γ and this is satisfied when $Q_3^{\alpha\beta} = Q_4^{\alpha\beta} = 0$.

On the second sector $M_8(\mathbb{C})$, the commutation relation with $a \in M_8(\mathbb{C}) \otimes \mathbb{I}_8$

$$A = \begin{pmatrix} M_1^{IJ} \otimes \mathbb{I}_8 & M_3^{IJ} \otimes \mathbb{I}_8 \\ M_4^{IJ} \otimes \mathbb{I}_8 & M_2^{IJ} \otimes \mathbb{I}_8 \end{pmatrix}, \quad M_i^{IJ} \in M_4(\mathbb{C}), \text{ for } i = 1, 2, 3, 4, \quad (4.8)$$

leads to

$$[a, \Gamma] = \begin{pmatrix} 0 & -2M_3^{IJ} \otimes \Gamma_8 \\ -2M_4^{IJ} \otimes \Gamma_8 & 0 \end{pmatrix} = 0 \Leftrightarrow M_3^{IJ} = M_4^{IJ} = 0, \quad (4.9)$$

It is important to notice that the fact that the algebra is not anymore acting only on internal indices causes a reduction not only in the quaternionic sector, as in the case of \mathcal{A}_F , but also the complex matrices part gets reduced.

In the next section we will see how this grand algebra gets reduced to the standard model algebra.

5 Spin Structure and the Grand Symmetry Breaking

In noncommutative geometry the topological information is encoded in the algebra, while the geometry is in the D operator[†]. In particular the Riemann-spin structure is encoded in the way the Dirac operator, which contains the gamma matrices acts on the Hilbert space. This is conceptually the passage from \mathcal{H}_F to \mathbf{H}_F in (2.18). Without a Dirac operator there is just an algebra which acts in an highly reducible way on a 128 dimensional Hilbert space. A spin structure means that the vectors in this Hilbert space transform in a particular representation under the “Lorentz” group. Since we are in the Euclidean case this is $\text{SO}(4)$. We now show that the presence of the free Dirac operator \not{D} causes a symmetry breaking phenomenon which reduces the algebra $\mathcal{A}_G \rightarrow \mathcal{A}_F$. The presence of the Majorana coupling and the σ field also appear at this stage.

The standard model can be considered as a “low energy” limit of the theory we present in this section. We will assume therefore that the quantities involved in D_F are small compared to the scales of the breaking described here. We will therefore deal with a Dirac operator composed of two parts. One is the usual \not{D} , and the other a finite dimensional matrix. The difference with the earlier case is that this finite matrix will not be acting only on the internal indices. Although the two parts of the operator are important at the same scale we will present them, and their action separately for clarity of the exposition. Therefore we will show first how the Majorana coupling at high scale gives the σ field, which breaks the left-right symmetry and gives two copies of the standard model, and then how the \not{D} part ensures that there are not two copies. We emphasize that in our view the two phenomena happen at the same scale, and that therefore there are never two copies of the standard model, or a $\text{SU}(8)$ symmetry separately.

5.1 The Majorana coupling and the σ field

The order-one condition, due to the Dirac operator without Majorana coupling, imposes a very similar reduction as in section 3.1, $\mathcal{A}'_G \rightarrow \mathcal{A}''_G$:

$$\mathcal{A}''_G = (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus (\mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})) \oplus (\mathbb{H}'_R \oplus \mathbb{H}'_L) \oplus (\mathbb{C}' \oplus \mathbb{M}'_3(\mathbb{C})) \quad (5.1)$$

[†]The metric aspects of the almost commutative geometry of the standard model have been investigated in [11, 49]

Consider now a finite part to the operator given by a matrix D_{ν_M} , (where M stands for Majorana) which has nonzero elements in correspondence to a Majorana coupling k_M , therefore the matrix $D_{IJ,CD}^{ab,\dot{a}\dot{b},\alpha\beta}$ has the following entries different from zero:

$$D_{00,01}^{rr,\dot{0}i,11} = D_{00,01}^{ll,\dot{0}i,11} = k_M \quad (5.2)$$

plus Hermitian conjugate. These entries connect right handed neutrinos among themselves, and are therefore Majorana couplings responsible for the seesaw mechanism. The novelty is that the matrix cannot be written in the usual form of $\gamma_5 \otimes D_F$, since it acts in a nontrivial way on the spin indices. It is not a matrix acting only on the internal indices, in this case there would be no contribution to the one form. Instead when one calculates the one form the commutation of this element with the elements of \mathcal{A}'_G gives a non-zero connection one-form.

Including in the Dirac operator also this Majorana coupling, we obtain from the first order condition the following further reduction,

$$(\mathbb{H}_R \oplus \mathbb{H}_L) \oplus (\mathbb{C} \oplus \mathbb{M}_3(\mathbb{C})) \rightarrow (\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})) \quad (5.3)$$

leading to the appearance of a first standard model algebra in \mathcal{A}''_G .

The same procedure, with the introduction of the same Majorana coupling between a left-handed neutrino and anti-neutrino:

$$D_{00,10}^{rr,\dot{0}i,33} = D_{00,10}^{ll,\dot{0}i,33} = k_M \quad (5.4)$$

plus Hermitian conjugate, can be used to reduce the remaining "primed algebra" to another standard model algebra,

$$(\mathbb{H}'_R \oplus \mathbb{H}'_L) \oplus (\mathbb{C}' \oplus \mathbb{M}'_3(\mathbb{C})) \rightarrow (\mathbb{C}' \oplus \mathbb{H}' \oplus \mathbb{M}'_3(\mathbb{C})) \quad (5.5)$$

obtaining two copies of the standard model,

$$\mathcal{A}'_{sm} = (\mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C})) \oplus (\mathbb{C}' \oplus \mathbb{H}' \oplus \mathbb{M}'_3(\mathbb{C})). \quad (5.6)$$

The crucial point now is that in the one-form connection $\sum_i A_i [D_{\nu_M}, B_i]$ appears twice the new scalar field σ in the position where k_M are. Note that at low energies, when the algebra \mathcal{A}'_{sm} is reduced to the usual algebra \mathcal{A}_{sm} , the Dirac operator becomes factorized as in 2.7 and the two σ fields coincide.

The Dirac operator, combined to this "double standard model algebra", \mathcal{A}'_{sm} , generates a new field in the correct position for the breaking of the left-right symmetry to the usual hypercharge-weak isospin group. The fact that algebra and Dirac operators are not diagonal in the spin indices is fundamental in the non-vanishing of the commutator, and hence in the appearance of the field. The nature of the Majorana mass of the neutrino is different from the Yukawa couplings of all other particles, and this fully justifies its appearance at low energy as a field, as done in [24]. It is natural to assume that the grand

symmetry is a high scale phenomenon, and therefore that the quantities involved are of the order of the Planck scale. This means that the natural scale for the right handed neutrino may be of the order of the unification scale, or even 10^{19} GeV, and this explains the large scale necessary for the see-saw mechanism.

We do not really envisage the presence of a phase in which a doubling of the standard model is present. At least not necessarily. The reason is that we consider the elimination of the doubling described next as happening concomitantly as high energy phenomena. Next we show how the doubling of the algebra is eliminated by a symmetry breaking similar to the Higgs mechanism.

5.2 Reduction to the simpler algebra and the spin structure

In this section we show the breaking of \mathcal{A}'_G defined in (4.5) to \mathcal{A}_F . In reality we should have considered the algebra \mathcal{A}'_{sm} of course. The problem is that in this case we must use a very cumbersome notation, which hides the phenomenon. We stress once more however that we envisage the breakings considered in this section and the previous as happening at the same scale.

Let us consider the free Dirac operator \not{D} . This operator only acts on the spin indices $a\dot{a}$ and is convenient to write this operator making explicit the block structure of the γ euclidean matrices:

$$\not{D} = \delta_{IJ}\delta_{CD}\delta^{\alpha\beta} \begin{pmatrix} \mathbb{O}_2 & \partial_\mu \sigma^\mu_{\dot{a}\dot{b}} \\ \partial_\mu \bar{\sigma}^\mu_{\dot{a}\dot{b}} & \mathbb{O}_2 \end{pmatrix}_{ab} \quad (5.7)$$

where $\sigma^\mu = (\mathbb{I}_2, -i\sigma_i)$ and $\bar{\sigma}^\mu = (\mathbb{I}_2, i\sigma_i)$ and σ_i are the three Pauli matrices.

Fluctuations of this Dirac operator are built out of elements of the algebra $A, B \in \mathcal{A}'_G$. Hence a generic one form is

$$\not{A} = A [\not{D}, B] \quad (5.8)$$

where $A, B \in \mathcal{A}'_G$.

$$A = \delta^{\dot{a}\dot{b}}\delta^{IJ} \begin{pmatrix} Q_1^{\alpha\beta} & 0_4 \\ 0_4 & Q_2^{\alpha\beta} \end{pmatrix}_{ab} \oplus \begin{pmatrix} M_1^{IJ} & 0_4 \\ 0_4 & M_2^{IJ} \end{pmatrix}_{\dot{a}\dot{b}} \delta^{\alpha\beta}\delta^{ab} \quad (5.9)$$

where $Q_{1,2}^{\alpha\beta} \in M_2(\mathbb{H})$ and $M_{1,2}^{IJ} \in M_4(\mathbb{C})$.

Rather than considering the role of J which exchanges the role of quaternionic and complex part of the algebra, in order not to render the notation too cumbersome, we will consider separately the contribution of the two sectors to the action. Since we are looking for the minimum of a potential, and we will find this to be zero, the procedure is equivalent to having the combination of the two actions.

Consider first:

$$\begin{aligned}
[\not{D}, B] &= \begin{pmatrix} 0_8 & \sigma^\mu \partial_\mu Q_2 + (\sigma^\mu Q_2 - Q_1 \sigma^\mu) \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu Q_1 + (\bar{\sigma}^\mu Q_1 - Q_2 \bar{\sigma}^\mu) \partial_\mu & 0_8 \end{pmatrix} \\
A[\not{D}, B] &= \begin{pmatrix} 0_8 & Q'_1 \sigma^\mu \partial_\mu Q_2 + Q'_1 (\sigma^\mu Q_2 - Q_1 \sigma^\mu) \partial_\mu \\ Q'_2 \bar{\sigma}^\mu \partial_\mu Q_1 + Q'_2 (\bar{\sigma}^\mu Q_1 - Q_2 \bar{\sigma}^\mu) \partial_\mu & 0_8 \end{pmatrix} \\
&= \begin{pmatrix} 0_8 & \sigma^\mu (B_\mu + \mathcal{A} \partial_\mu) \\ \bar{\sigma}^\mu (\tilde{B}_\mu + \tilde{\mathcal{A}} \partial_\mu) & 0_8 \end{pmatrix}
\end{aligned} \tag{5.10}$$

where we defined the quantities:

$$\begin{aligned}
B_\mu &\equiv Q'_1 \partial_\mu Q_2 \quad , \quad \mathcal{A} \equiv Q'_1 (Q_2 - Q_1) \\
\tilde{B}_\mu &\equiv Q'_2 \partial_\mu Q_1 \quad , \quad \tilde{\mathcal{A}} \equiv Q'_2 (Q_1 - Q_2) .
\end{aligned} \tag{5.11}$$

We can write the fluctuated Dirac operator:

$$\begin{aligned}
\not{D}_A &= \not{D} + \not{A} \\
&= \begin{pmatrix} 0_8 & \sigma^\mu (\partial_\mu + B_\mu + \mathcal{A} \partial_\mu) \\ \bar{\sigma}^\mu (\partial_\mu + \tilde{B}_\mu + \tilde{\mathcal{A}} \partial_\mu) & 0_8 \end{pmatrix} \\
&\equiv \begin{pmatrix} 0_8 & \sigma^\mu (B_\mu + A \partial_\mu) \\ \bar{\sigma}^\mu (\tilde{B}_\mu + \tilde{A} \partial_\mu) & 0_8 \end{pmatrix}
\end{aligned} \tag{5.12}$$

where we have redefined $A = 1 + \mathcal{A}$ and $\tilde{A} = 1 + \tilde{\mathcal{A}}$.

It is easy to check that the self-adjoint condition for the Dirac operator constrains \tilde{B}_μ and $\tilde{\mathcal{A}}$ to be respectively equal to B_μ^\dagger and A^\dagger .

In view of the calculation of the full spectral action let us first calculate the square of this Dirac operator and trace it over the spin indices. The rationale behind this is that we are considering the theory at a very high energy, for which the internal degrees of freedom are frozen and only spacetime counts. We will then perform a reduced spectral action calculation to see the potential of this theory. In the following by tr we mean a trace over only the a and \dot{a} indices.

$$\begin{aligned}
\text{tr}(\not{D}_A^2) &= \text{tr}[\sigma^\mu \bar{\sigma}^\nu (A \partial_\mu + B_\mu)(\tilde{A} \partial_\nu + \tilde{B}_\nu) + \sigma^\nu \bar{\sigma}^\mu (\tilde{A} \partial_\nu + \tilde{B}_\nu)(A \partial_\mu + B_\mu)] \\
&= 2\delta^{\mu\nu}[(A \partial_\mu + B_\mu)(\tilde{A} \partial_\nu + \tilde{B}_\nu) + (\tilde{A} \partial_\nu + \tilde{B}_\nu)(A \partial_\mu + B_\mu)] \\
&= 2[(A \tilde{A} + \tilde{A} A) \delta^{\mu\nu} \partial_\mu \partial_\nu + \\
&\quad + (A \partial^\mu \tilde{A} + \tilde{A} \partial^\mu A + A \tilde{B}^\mu + \tilde{A} B^\mu + B^\mu \tilde{A} + \tilde{B}^\mu A) \partial_\mu + \\
&\quad + (A \partial_\mu \tilde{B}^\mu + \tilde{A} \partial_\mu B^\mu + B_\mu \tilde{B}^\mu + \tilde{B}_\mu B^\mu)]
\end{aligned} \tag{5.13}$$

by setting

$$\begin{aligned}
g^{\mu\nu} &= (A \tilde{A} + \tilde{A} A) \delta^{\mu\nu} \\
\alpha^\mu &= A \partial^\mu \tilde{A} + \tilde{A} \partial^\mu A + A \tilde{B}^\mu + \tilde{A} B^\mu + B^\mu \tilde{A} + \tilde{B}^\mu A \\
\beta &= A \partial_\mu \tilde{B}^\mu + \tilde{A} \partial_\mu B^\mu + B_\mu \tilde{B}^\mu + \tilde{B}_\mu B^\mu
\end{aligned} \tag{5.14}$$

the spectral action is put into the elliptic operator form,

$$\text{tr}(\not{D}_A^2) = -(g^{\mu\nu}\partial_\mu\partial_\nu + \alpha^\mu\partial_\mu + \beta) \quad (5.15)$$

Applying the usual definitions of (3.13) the elliptic operator is rewritten in the form,

$$\text{tr}(\not{D}_A^2) = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \quad (5.16)$$

From (3.14) it can be seen that the potential (i.e. the terms which do not depend on derivatives) is a positive function of E involving a linear combination of $\text{Tr } E$ and $\text{Tr } E^2$ in a linear combination with positive coefficients. On the other side an explicit calculation with a symbolic manipulation programme of the eigenvalues of E show that they are positive definite. They are ratios of polynomials involving sums of squares of the elements of \mathbf{A} and \mathbf{B} of (5.9). The minimum of this potential is therefore attained when $E = 0$. This corresponds to the solution

$$A = \tilde{A} = 1 \iff \mathcal{A} = \tilde{\mathcal{A}} = 0 \iff Q_1 = Q_2. \quad (5.17)$$

In this case in fact we also have $\tilde{B}^\mu = B^\mu$, $g^{\mu\nu} = 2\delta^{\mu\nu}$, $g_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}$, $\Gamma_{\mu\nu}^\sigma = 0$, and

$$\begin{aligned} \beta &= 2\partial_\mu B^\mu + 2B_\mu B^\mu \\ g^{\mu\nu}\partial_\nu\omega_\mu &= 2 \cdot \frac{1}{2} \cdot \frac{1}{2}\partial_\mu(4B^\mu) = 2\partial_\mu B^\mu \\ g^{\mu\nu}\omega_\mu\omega_\nu &= 2\left(\frac{1}{2} \cdot \frac{1}{2} \cdot 4B_\mu\right)\left(\frac{1}{2} \cdot \frac{1}{2} \cdot 4B^\mu\right) = 2B_\mu B^\mu \end{aligned} \quad (5.18)$$

which leads to $E = 0$.

The important result is that this “minimum condition” reduces $[M_2(\mathbb{H}) \oplus M_2(\mathbb{H})]$ to $M_2(\mathbb{H})$. This is symmetry breaking phenomenon similar to the Higgs breaking, that is, the minimum of the potential is not invariant with respect to the full group, but only with respect to a subgroup.

The same procedure can be applied to the second sector:

$$[\not{D}, \mathbf{B}] = \begin{pmatrix} 0 & (\sigma^\mu\partial_\mu M - M\sigma^\mu\partial_\mu) \\ (\bar{\sigma}^\mu\partial_\mu M - M\bar{\sigma}^\mu\partial_\mu) & 0 \end{pmatrix} \quad (5.19)$$

with $M = \begin{pmatrix} M_1^{IJ} & 0 \\ 0 & M_2^{IJ} \end{pmatrix}_{\dot{a}\dot{b}}$ and $\sigma^\mu = (\sigma^\mu)_{\dot{a}\dot{b}}$. The full one form is:

$$\mathbf{A}[\not{D}, \mathbf{B}] = \begin{pmatrix} 0 & M'(\sigma^\mu\partial_\mu M - M\sigma^\mu\partial_\mu) \\ M'(\bar{\sigma}^\mu\partial_\mu M - M\bar{\sigma}^\mu\partial_\mu) & 0 \end{pmatrix} \quad (5.20)$$

We can write \not{D}_A ,

$$\not{D}_A = \not{D} + \begin{pmatrix} 0 & M'\sigma^\mu\partial_\mu M + M'(\sigma^\mu M - M\sigma^\mu)\partial_\mu \\ M'\bar{\sigma}^\mu\partial_\mu M + M'(\bar{\sigma}^\mu M - M\bar{\sigma}^\mu)\partial_\mu & 0 \end{pmatrix} \quad (5.21)$$

In the previous case we have seen that the potential E in the spectral action has a minimum when $\mathcal{A} = 0$ that is when the matrix in front of ∂_μ in the 1-form connection $a[\not{\partial}, b]$ is equal to zero. In this other case the same condition is verified when the matrix M commute with σ that is when M is diagonal in the $\dot{a}\dot{b}$ indices,

$$M = M^{IJ}\delta_{\dot{a}\dot{b}} \quad (5.22)$$

Therefore also in the second sector the “minimum condition” reduces $[M_4(\mathbb{C}) \oplus M_4(\mathbb{C})]$ to $M_4(\mathbb{C})$.

There remain the fluctuations of the connection around the minimum, these can be interpreted as the spin connection plus the gauge connection. In this sense they appear as some sort of Higgs field of a spontaneously broken symmetry, where the broken symmetry is the $O(4)$ in the spin representation times the gauge group, and the unbroken group the unitary elements of the algebra \mathcal{A}_G .

6 Conclusions and Outlook

In this paper we have introduced a higher level of symmetry in the noncommutative geometry standard model. It is based on the fact that the almost commutative geometry should be a manifold. The higher symmetry has two main features, on one side it explains the presence of the σ field necessary for a correct fit of the mass of the Higgs, and on the other shows that the spin structure is the result of a symmetry breaking. The results presents here, as is common in this model, are crucially depending on the euclidean structure of the theory. This is particularly important as far the role of chirality and the doubling of the degrees of freedom is concerned. A Wick rotation is far from simple in this context, and on the other side the construction of a Minkowskian noncommutative geometry is yet to come (for recent works see [50, 51]).

The presence of this grand symmetry will have also phenomenological consequences which should be investigated. The breaking mechanism we described in the previous section is just barely sketched, we only looked at the group structure. A more punctual analysis should reveal more structure, and possibly alter the running of the constants at high energy.

For the moment we can only speculate. One of the problems of the spectral action in its present form is the fact that it requires unification of the three gauge couplings at a single scale, Λ , and physical predictions are based on the value of this scale. It is known experimentally that in the absence of new physics the three constants do not meet in a single point, but the three lines form an elongated triangle spanning nearly three orders of magnitude. On the other side in the spectral action is not clear what would happen after this point, if one consider scales higher than Λ , i.e. earlier epochs. For a theory dealing with the unification of gauge theory and gravity a more natural scale is the Planck scale. An unification of the coupling constants at the Planck scale in the form of a pole has been

considered [52, 53], but it requires new fermions. In the case at hand the “new physics” is in the form of a different structure which mixes spacetime spin and gauge degrees of freedom. This might have consequences for the interactions, and hence for the running of the various quantities, as well.

There are other issues that one should investigate. For example we considered for algebras of the kind $M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})$ only the case $a = 2, 4$. What about the case $a = 3$? Could it be related to the number of generations? What about the next level up? Is the Grand symmetry an effective theory as well? It is likely that the next symmetry will not have an almost commutative geometry, but will require a more profound alteration of spacetime.

Acknowledgments We would like to thank M.A. Kurkov for discussions. F. Lizzi acknowledges support by CUR Generalitat de Catalunya under project FPA2010-20807. P.M. warmly thanks A. Roche for her constant support.

References

- [1] A. Connes, *Noncommutative Geometry*, Academic Press, 1984.
- [2] G. Landi, *An Introduction to Noncommutative Spaces and their Geometries*, *Springer Lecture Notes in Physics 51*, Springer Verlag (Berlin Heidelberg) 1997. arXiv:hep-th/9701078.
- [3] J.M. Gracia-Bondia, J.C. Varilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhauser, 2000.
- [4] A. Connes, M. Marcolli, “Noncommutative Geometry, Quantum Fields and Motives”, AMS 2007;
- [5] A. Connes and J. Lott, “Particle Models And Noncommutative Geometry (expanded Version),” Nucl. Phys. Proc. Suppl. **18B** (1991) 29.
- [6] T. Schucker, “Forces from Connes’ geometry,” Lect. Notes Phys. **659** (2005) 285 [hep-th/0111236].
- [7] A. H. Chamseddine, A. Connes and M. Marcolli, “Gravity and the standard model with neutrino mixing,” Adv. Theor. Math. Phys. **11** (2007) 991 [arXiv:hep-th/0610241].
- [8] K. van den Dungen and W. D. van Suijlekom, “Particle Physics from Almost Commutative Spacetimes,” arXiv:1204.0328 [hep-th].

- [9] A. Connes, “On the spectral characterization of manifolds,” *J. Noncommutative Geometry*, **7** 1, 2013. arXiv:0810.2088 [math.OA].
- [10] A. H. Chamseddine and A. Connes, “Noncommutative Geometry as a Framework for Unification of all Fundamental Interactions including Gravity. Part I,” *Fortsch. Phys.* **58** (2010) 553 [arXiv:1004.0464 [hep-th]].
- [11] A. H. Chamseddine and A. Connes, “The spectral action principle,” *Commun. Math. Phys.* **186**, 731 (1997) [arXiv:hep-th/9606001].
- [12] K. Fujikawa, “Comment On Chiral And Conformal Anomalies,” *Phys. Rev. Lett.* **44**, 1733 (1980).
- [13] A. A. Andrianov and L. Bonora, “Finite-Mode Regularization Of The Fermion Functional Integral,” *Nucl. Phys. B* **233**, 232 (1984).
- [14] A. A. Andrianov and L. Bonora, “Finite Mode Regularization Of The Fermion Functional Integral. 2,” *Nucl. Phys. B* **233**, 247 (1984).
- [15] J. Polchinski, “Renormalization and Effective Lagrangians,” *Nucl. Phys. B* **231** (1984) 269.
- [16] F. Lizzi and P. Vitale, “Gauge and Poincaré Invariant Regularization and Hopf Symmetries,” *Mod. Phys. Lett. A* **27** (2012) 1250097 [arXiv:1202.1190 [hep-th]].
- [17] A. A. Andrianov and F. Lizzi, “Bosonic Spectral Action Induced from Anomaly Cancellation,” *JHEP* **1005** (2010) 057 [arXiv:1001.2036 [hep-th]].
- [18] A. A. Andrianov, M. A. Kurkov and F. Lizzi, “Spectral action, Weyl anomaly and the Higgs-Dilaton potential,” *JHEP* **1110** (2011) 001 [arXiv:1106.3263 [hep-th]].
- [19] M. A. Kurkov and F. Lizzi, “Higgs-Dilaton Lagrangian from Spectral Regularization,” arXiv:1210.2663 [hep-th].
- [20] J. W. Barrett, “A Lorentzian version of the non-commutative geometry of the standard model of particle physics,” *J. Math. Phys.* **48** (2007) 012303 [hep-th/0608221].
- [21] W. Nelson and M. Sakellariadou, “Natural inflation mechanism in asymptotic non-commutative geometry,” *Phys. Lett. B* **680** (2009) 263 [arXiv:0903.1520 [hep-th]].
- [22] M. Sakellariadou, “Cosmological consequences of the noncommutative spectral geometry as an approach to unification,” *J. Phys. Conf. Ser.* **283** (2011) 012031 [arXiv:1010.4518 [hep-th]].
- [23] M. Marcolli and E. Pierpaoli, “Early Universe models from Noncommutative Geometry,” *Adv. Theor. Math. Phys.* **14** (2010) [arXiv:0908.3683 [hep-th]].

- [24] A. H. Chamseddine and A. Connes, “Resilience of the Spectral Standard Model,” arXiv:1208.1030 [hep-ph].
- [25] R. Wulkenhaar, “The Standard model within nonassociative geometry,” Phys. Lett. B **390** (1997) 119 [hep-th/9607096].
- [26] S. Farnsworth and L. Boyle, “Non-Associative Geometry and the Spectral Action Principle,” arXiv:1303.1782 [hep-th].
- [27] F. Lizzi, G. Mangano, G. Miele and G. Sparano, “Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories,” Phys. Rev. D **55** (1997) 6357 [hep-th/9610035].
- [28] J. M. Gracia-Bondia, B. Iochum and T. Schucker, “The Standard model in noncommutative geometry and fermion doubling,” Phys. Lett. B **416** (1998) 123 [hep-th/9709145].
- [29] F. Lizzi, G. Mangano, G. Miele and G. Sparano, “Mirror fermions in noncommutative geometry,” Mod. Phys. Lett. A **13** (1998) 231 [hep-th/9704184].
- [30] A. Connes. “Gravity coupled with matter and the foundations of noncommutative geometry”, Commun. Math. Phys. **182** (1996) 155–176.
- [31] T. van den Broek and W. D. van Suijlekom, “Supersymmetric QCD from noncommutative geometry,” Phys. Lett. B **699** (2011) 119.
- [32] M.A. Rieffel, “Morita equivalence for operator algebras”, in *Operator Algebras and Applications* (R. V. Kadison, ed.), 285-298, Proc. Symp. Pure Math. 38, Amer. Math. Soc., Providence, 1982.
- [33] J. C. Pati and A. Salam, “Lepton Number as the Fourth Color,” Phys. Rev. D **10** (1974) 275 [Erratum-ibid. D **11** (1975) 703].
- [34] F. Lizzi, G. Mangano, G. Miele and G. Sparano, “Constraints on unified gauge theories from noncommutative geometry,” Mod. Phys. Lett. A **11** (1996) 2561 [hep-th/9603095].
- [35] M. Dubois-Violette, J. Madore and R. Kerner, “Classical Bosons In A Noncommutative Geometry,” Class. Quant. Grav. **6** (1989) 1709. M. Dubois-Violette, R. Kerner and J. Madore, “Noncommutative Differential Geometry of Matrix Algebras,” J. Math. Phys. **31** (1990) 316.
- [36] P. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, Publish or Perish, 1984.
- [37] D. V. Vassilevich, “Heat kernel expansion: User’s manual,” Phys. Rept. **388** (2003) 279 [hep-th/0306138].

- [38] G. Aad *et al.* [ATLAS Collaboration], “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” *Phys. Lett. B* **716** (2012) 1 [arXiv:1207.7214 [hep-ex]]. S. Chatrchyan *et al.* [CMS Collaboration], “Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC,” *Phys. Lett. B* **716** (2012) 30 [arXiv:1207.7235 [hep-ex]].
- [39] C. A. Stephan, “New Scalar Fields in Noncommutative Geometry,” *Phys. Rev. D* **79** (2009) 065013 [arXiv:0901.4676 [hep-th]].
- [40] I. Pris and T. Schücker, “Noncommutative geometry beyond the standard model,” *J. Math. Phys.* **38** (1997) 2255 [hep-th/9604115].
- [41] M. Paschke, F. Scheck and A. Sitarz, “Can (noncommutative) geometry accommodate leptoquarks?,” *Phys. Rev. D* **59** (1999) 035003 [hep-th/9709009].
- [42] T. Schucker and S. Zouzou, “Spectral action beyond the standard model,” hep-th/0109124.
- [43] C. A. Stephan, “Almost-commutative geometries beyond the standard model,” *J. Phys. A* **39** (2006) 9657 [hep-th/0509213].
- [44] C. A. Stephan, “Almost-commutative geometries beyond the standard model. II. New Colours,” *J. Phys. A* **40** (2007) 9941 [arXiv:0706.0595 [hep-th]].
- [45] R. Squellari and C. A. Stephan, “Almost-Commutative Geometries Beyond the Standard Model. III. Vector Doublets,” *J. Phys. A* **40** (2007) 10685 [arXiv:0706.3112 [hep-th]].
- [46] C. A. Stephan, “Beyond the Standard Model: A Noncommutative Approach,” arXiv:0905.0997 [hep-ph].
- [47] C. -S. Chen and Y. Tang, “Vacuum stability, neutrinos, and dark matter,” *JHEP* **1204**, 019 (2012) [arXiv:1202.5717 [hep-ph]].
- [48] M. Gonderinger, Y. Li, H. Patel and M. J. Ramsey-Musolf, “Vacuum Stability, Perturbativity, and Scalar Singlet Dark Matter,” *JHEP* **1001** (2010) 053 [arXiv:0910.3167 [hep-ph]].
- [49] P. Martinetti and R. Wulkenhaar, “Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry,” *J. Math. Phys.* **43** (2002) 182 [hep-th/0104108].
- [50] M. Paschke, R. Verch and , “Local covariant quantum field theory over spectral geometries,” *Class. Quant. Grav.* **21** (2004) 5299 [gr-qc/0405057].
- [51] N. Franco, “Lorentzian approach to noncommutative geometry,” arXiv:1108.0592 [math-ph].

- [52] L. Maiani, G. Parisi and R. Petronzio, “Bounds on the Number and Masses of Quarks and Leptons,” Nucl. Phys. B **136**, 115 (1978).
- [53] A. A. Andrianov, D. Espriu, M. A. Kurkov and F. Lizzi, “Universal Landau Pole,” arXiv:1302.4321 [hep-th].